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DYNAMIC SNAP-THROUGH OF AN ELASTIC–PLASTIC SIMPLE SHALLOW TRUSS†

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Abstract—The effect of plastic deformation on the dynamic buckling of shallow structures is studied for a simple truss consisting of two identical bars with a mass attached to the middle hinge. The material of the truss is considered to be elastic–perfectly plastic with equal yield limits in tension and compression. The load is applied at the middle hinge. Two types of loading are included, namely, step loading and impulsive loading. It is found that for materials with low level of yield stress, the dynamic buckling load increases with the yield stress. However, for materials with a high level of yield stress, the dynamic buckling phenomenon is dominated by the elastic action. When the applied load is sufficiently large, the backward snap-through of the truss may be prevented by the tensile yielding present in the post-buckling deformation of the truss.

NOMENCLATURE

A cross-sectional area of the bar
a half span of the truss
E Young's modulus of the bar
f_1 a function, equation (13)
f_2 a function, equation (18)
f_3 a function, equation (30)
f_4 a function, equation (32)
f_5 a function, equation (36)
$H(t)$ Heaviside step function of time
$h(t)$ height of the truss at time t
h_y height of the truss at the beginning of compressive yielding
h_0 initial height of the truss
M attached mass at the middle hinge
P_1 magnitude of the impulsive load
P_2 magnitude of the step load
p dimensionless step load, equation (6)
p_c critical load for snap-through
$\bar{p}_c$ critical load for snap-through during compressive yielding, equation (59)
p^* a root of equation (36)
t time
V initial downward velocity of the middle hinge
v initial velocity parameter, equation (6)
v_c critical initial velocity for snap-through
v_c^* critical initial velocity for backward snap-through
X $= v^2+2p$
y $= h/h_0$
$\dot{y}$ $= \dfrac{dy}{d\tau}$

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$\ddot{y} \quad = \dfrac{d^2y}{d\tau^2}$
y_{c1} saddle point height, equation (15)
y_{c2} saddle point height, equation (31)
y_m minimum height
y_{yc} height at the beginning of compressive yielding during forward motion
y^*_{yc} height at the beginning of compressive yielding during backward motion
y_{yt} height at the beginning of tensile yielding
α yield stress parameter, equation (6)
$\delta(t)$ Dirac delta function of time
ε strain of the bar
ε_0 total strain at $y = 0$
ε_m total strain at $y = y_m$
ε_y strain at the beginning of compressive yielding
θ inclination of the bar at time t
θ_0 initial inclination of the bar
σ stress in the bar
σ_y yield stress of the bar
τ dimensionless time, equation (6)

1. INTRODUCTION

DYNAMIC buckling of shallow structures is often accompanied by plastic deformation. Such phenomenon was observed by Humphreys [1] and Lock *et al.* [2] in their experiments on dynamic snap-through of shallow arches and spherical caps. Due to the presence of plastic flow, the dynamic buckling load is reduced and backward snapping may be prevented. Thus, dynamic buckling can be analyzed in a more realistic manner by taking plasticity into consideration.

The mathematical treatment in problems of dynamics of elastic–plastic structures is usually complicated due to the loading and unloading processes involved. Such complexity is influenced by the degree of freedom of the structure and the adopted constitutive law of the material. In order to make our study of the effect of plasticity on the dynamic buckling possible, we shall deal with a simple structure under a simple elastic–plastic constitutive law. The structure considered here is a symmetrical weightless shallow truss with a mass attached at the middle hinge as shown in Fig. 1. Motion of the truss is caused by a downward force applied at the middle hinge. Two types of loading are included in this investigation, namely, the Heaviside step loading and the Dirac impulsive loading. The material of the truss is considered to be elastic for stresses within yield limits and to be perfectly plastic

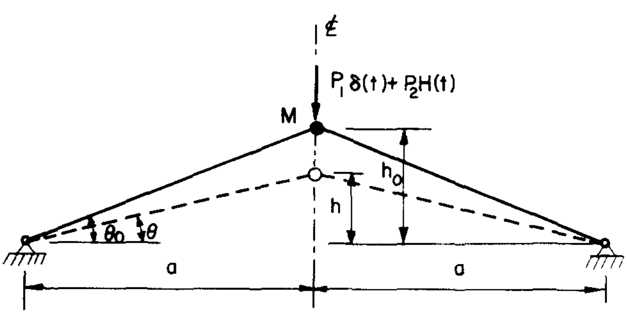

FIG. 1. Geometry of the problem.

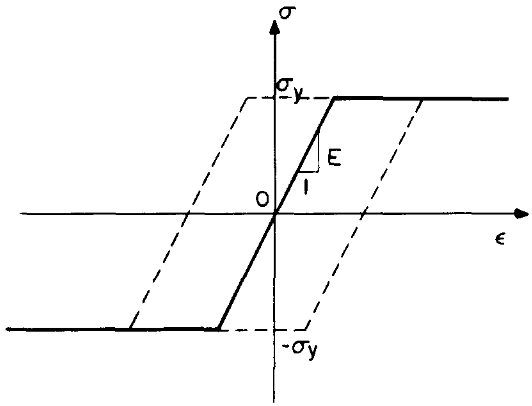

FIG. 2. Stress–strain relation for elastic–perfectly plastic materials.

for stresses equal to the yield stress. The stress–strain relation is demonstrated graphically in Fig. 2. The purpose of this study is to find useful information on the one-dimensional dynamic buckling phenomenon, hopefully applicable to some more complicated elastic–plastic structures such as perfectly symmetrical arches or axisymmetrical spherical caps under perfectly symmetrical loading conditions.

2. FORMULATION OF THE PROBLEM

A symmetrical pin-jointed weightless shallow truss with span $2a$ and height h_0 is attached to a mass M at the middle hinge as shown in Fig. 1. An impulsive load $P_1\delta(t)$ and a step load $P_2H(t)$ are applied simultaneously at the middle hinge, where $\delta(t)$ and $H(t)$ are the Dirac delta function and the Heaviside step function of time, respectively. The downward velocity of the middle hinge at $t = 0$ caused by the impulsive load is denoted by V. At any time t the height of the truss is $h(t)$. Since the truss is assumed to be shallow, the approximations $\theta \simeq h/a$ and $\theta_0 \simeq h_0/a$ are valid. The strain of the bar is approximately

$$\varepsilon = \frac{1}{2a^2}(h^2 - h_0^2). \tag{1}$$

We shall assume that the bars are sufficiently rigid against bending so that they remain straight during deformation. For $t > 0$, the stress in each bar is

$$\sigma = -\frac{a}{2Ah}\left(P_2 + M\frac{\mathrm{d}^2h}{\mathrm{d}t^2}\right) \tag{2}$$

where A is the cross-sectional area of the bar which is considered to be uniform. The second term in the parenthesis is the D'Alembert force caused by the acceleration of the attached mass.

The material of the truss is considered to be elastic–perfectly plastic with the magnitude of yield stress σ_y in both tension and compression. Within the yield limits, the stress–strain relation is elastic with Young's modulus E. The stress–strain relation is shown graphically in Fig. 2.

In the early stage of deformation, the stress in the bar is compressive and within the yield limit, thus

$$\sigma = E\varepsilon. \tag{3}$$

Equation (3) holds as long as $\sigma > -\sigma_y$. When $\sigma = -\sigma_y$, compressive yielding begins. If we denote the height of the truss and the strain of the bar at the instant of initiation of compressive yielding by h_y and ε_y respectively, we have

$$\varepsilon_y = \frac{1}{2a^2}(h_y^2 - h_0^2) \tag{4}$$

and

$$-\sigma_y = E\varepsilon_y. \tag{5}$$

Let us introduce the following dimensionless quantities:

$$\begin{aligned} &y = h/h_0, \qquad y_{yc} = h_y/h_0, \qquad p = P_2 a^3/(h_0^3 EA), \\ &\tau = t[Ah_0^2E/(Ma^3)]^{\frac{1}{2}}, \qquad \alpha^2 = 2\sigma_y a^2/(h_0^2 E), \\ &v = V[Ma^3/(Ah_0^4E)]^{\frac{1}{2}}. \end{aligned} \tag{6}$$

From equations (1), (2) and (3), we obtain the following equation of motion before compressive yielding:

$$\ddot{y} + y(y^2 - 1) + p = 0, \tag{7}$$

where $(\dot{\ }) \equiv \mathrm{d}/\mathrm{d}\tau\,(\ )$. The initial conditions expressed in dimensionless quantities are

$$y(0) = 1 \tag{8}$$

and

$$\dot{y}(0) = -v. \tag{9}$$

From equations (4) and (5), the height of the truss at the beginning of compressive yielding can be determined as

$$y_{yc} = (1 - \alpha^2)^{\frac{1}{2}}. \tag{10}$$

Therefore, when $\alpha \geq 1$, there is no possibility for compressive yielding.

Equations (7), (8) and (9) define an initial value problem for $y(\tau)$. The solution involves elliptic functions and it is difficult to use it in our buckling analysis. In this paper, instead of considering y as a function of τ, we shall treat $\dot{y}$ as a function of y and analyze the motion of the truss by means of the phase–plane diagram. The advantage of the phase–plane diagram analysis is that the solution of $\dot{y}(y)$ can be obtained directly from the equation of conservation of energy which is usually very simple in expression. The energy equation for the motion of the structure before compressive yielding can be obtained by multiplying both sides of equation (7) by $\dot{y}\,\mathrm{d}\tau$ and integrating from $\tau = 0$ to an arbitrary τ, while taking into account the initial conditions of equations (8) and (9). We have

$$\tfrac{1}{2}(\dot{y}^2 - v^2) + \tfrac{1}{4}(1 - y^2)^2 - p(1 - y) = 0. \tag{11}$$

The first term of equation (11) stands for the change of the kinetic energy; the second term represents the strain energy stored elastically in the structure and the last term is the work done by the step load. Hence equation (11) represents the conservation of energy in dimensionless form. If we plot $\dot{y}$ against y according to equation (11) by taking p and v as parameters, we obtain the trajectory curves in the phase–plane diagram which are shown in Fig. 3. The snap-through behavior can then be easily studied from the character of the trajectory curves. For any sufficiently small values of p and v, the trajectory consists of two non-intersecting closed curves which are shown by solid lines in Fig. 3. These

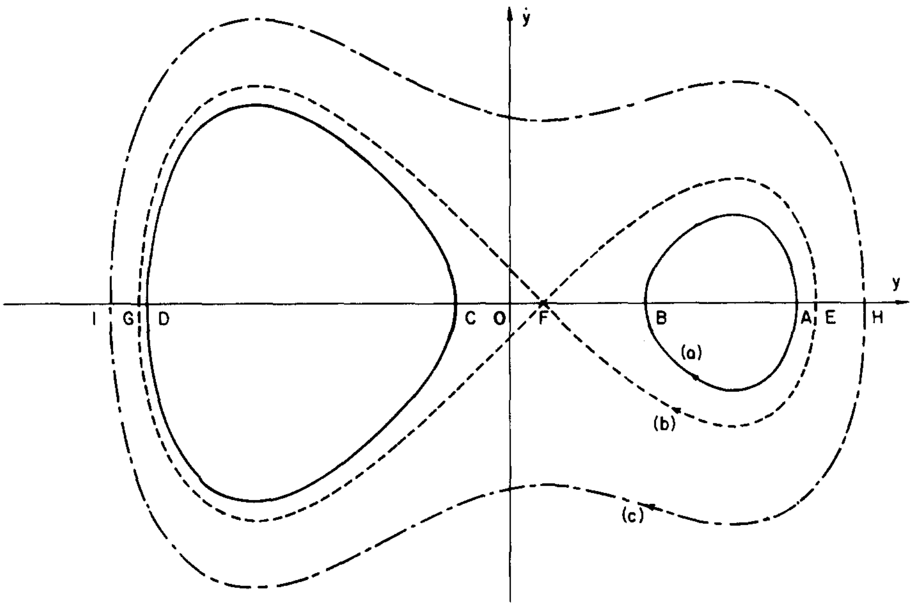

FIG. 3. Phase–plane diagram for elastic deformation of the truss.

curves intersect the y-axis at four points A, B, C and D corresponding to the four real roots of the algebraic equation

$$\tfrac{1}{4}(1-y^2)^2 - p(1-y) = \tfrac{1}{2}v^2. \tag{12}$$

Under the initial condition equation (8), the actual motion of the truss follows the right closed curve and the left one can never be reached. Under a certain critical condition of p and v, these four roots of y in equation (12) degenerate into three roots which are shown by points E, F and G in Fig. 3. The root at the point F is a double root. The condition for the presence of a double root in equation (12) is

$$f_1 = 8X^3 - 4X^2 - 36p^2X + p^2(16+27p^2) = 0, \tag{13}$$

where

$$X = v^2 + 2p. \tag{14}$$

The double root is found to be

$$y_{c1} = \frac{2p(2-3v^2-6p)}{2v^2+4p-9p^2}, \tag{15}$$

where p and v satisfy equation (13). The time required to reach the saddle point is infinite. For large values of p or v, equation (12) has only two real roots at points H and I in Fig. 3. In this case, if the stress in the truss remains within the yield limits, there is only one closed trajectory shown by the dashed line. Hence equation (13) is the condition for a discontinuous change of the character of trajectories in the phase–plane diagram and will be considered as the critical condition for dynamic snap-through. This definition of the dynamic snap-through of structures with one degree of freedom by the change of trajectory regions in the phase–plane from one side of the separatrix curve to the other was also adopted by Nachbar and Huang [3] in their study of a similar problem. Due to the effect of the plastic deformation, the actual trajectories may be different from those shown in Fig. 3. We shall discuss the snap-through problem with plastic deformations in the following sections.

3. CASE A: HIGH YIELD STRESS ($\alpha \geq 1$)

When $\alpha \geq 1$, there is no compressive yielding. If $f_1 \leq 0$, equation (12) possesses four real roots and the truss will oscillate elastically without snap-through. On the other hand, if $f_1 > 0$, the truss will deform to the snapped position. Furthermore, after the snap-through, if the tensile stress in bars reaches the yield stress σ_y, tensile yielding will occur. When tensile yielding begins, we have

$$\sigma_y = E\varepsilon_y. \tag{16}$$

At this moment, the height of the truss can be found from equations (4) and (16). It is

$$y_{yt} = -(1+\alpha^2)^{\frac{1}{2}}. \tag{17}$$

The velocity at $y = y_{yt}$ is denoted by $\dot{y}_{yt}$. From equation (22) we have

$$\dot{y}_{yt}^2 = v^2 - \tfrac{1}{2}\alpha^4 + 2p[1+(1+\alpha^2)^{\frac{1}{2}}] = f_2. \tag{18}$$

If $f_2 \leq 0$, there will be no tensile yielding after snap-through and the truss will snap back and forth elastically. On the other hand, if $f_1 > 0$ and $f_2 > 0$, tensile yielding will occur after snap-through.

During tensile yielding, the stress in the bar remains a constant value σ_y. The equation of motion can be obtained from equation (2) by letting $\sigma = \sigma_y$. Thus we obtain

$$\ddot{y} + \alpha^2 y + p = 0. \tag{19}$$

The energy equation can be derived from equation (19) by integration and utilization of the initial conditions, equations (17) and (18),

$$\dot{y}^2 = v^2 + \alpha^2(1-y^2) + 2p(1-y) + \tfrac{1}{2}\alpha^4. \tag{20}$$

The truss will move downward continuously until $\dot{y} = 0$. The minimum height of the truss, y_m, can be determined from equation (20) by setting $\dot{y} = 0$. Accordingly,

$$y_m = -\frac{p}{\alpha^2} - \left[\left(\frac{p}{\alpha^2}+1\right)^2 + \frac{v^2}{\alpha^2} + \frac{\alpha^2}{2}\right]^{\frac{1}{2}}. \tag{21}$$

At this height, the total strain is

$$\varepsilon_m = \frac{h_0^2}{2a^2}(y_m^2 - 1). \tag{22}$$

After the truss reaches the minimum point, it will move upward. In the upward motion, the stress–strain relation becomes elastic and satisfies the equation

$$\sigma-\sigma_y = E(\varepsilon-\varepsilon_m). \tag{23}$$

From equations (1), (2), (22) and (23), we can derive the following equation of motion for the upward motion of the truss:

$$\ddot{y}+y(y^2+\alpha^2-y_m^2)+p = 0. \tag{24}$$

The corresponding energy equation is

$$\dot{y}^2+\tfrac{1}{2}(y^2-y_m^2)^2+\alpha^2(y^2-y_m^2)+2p(y-y_m) = 0. \tag{25}$$

Note that equation (23) can also be written as

$$\sigma-\sigma_y = \frac{Eh_0^2}{2a^2}(y^2-y_m^2). \tag{26}$$

Hence, when the stress in the bar reaches the compressive yield stress during the backward motion, the truss height y_{yc}^* can be found from equation (26) by setting $\sigma = -\sigma_y$. Thus we have

$$y_{yc}^* = -(y_m^2-2\alpha^2)^{\frac{1}{2}}. \tag{27}$$

When $y = y_{yc}^*$, the velocity is $\dot{y}_{yc}^*$ which can be determined from equations (25) and (27),

$$\dot{y}_{yc}^* = \{2p[y_m+(y_m^2-2\alpha^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}. \tag{28}$$

Since y_m is negative, $\dot{y}_{yc}^*$ is complex and the truss will move downward before compression yielding. The maximum height in the post-buckling oscillation, y_n, can be determined from the following equation

$$(y_n^2-y_m^2+2\alpha^2)(y_n+y_m)+4p = 0. \tag{29}$$

If equation (29) possesses three real roots, there is no snap-through in the upward motion. On the other hand, if equation (29) possesses only one real root, there is a backward snap-through in the upward motion of the truss. Thus the critical condition for snap-through is also the condition for equation (29) possessing a double root. This condition can be found as

$$f_3 = 8py_m^3+\alpha^4y_m^2-18\alpha^2y_mp-2\alpha^6-27p^2 = 0, \tag{30}$$

where y_m is given by equation (21). The saddle point is located at

$$y_{c2} = \frac{1}{2y_m}[(\alpha^4-12py_m+8\alpha^2y_m^2+4y_m^4)^{\frac{1}{2}}-\alpha^2], \tag{31}$$

where p satisfies equation (30). If $f_3 < 0$, equation (29) has only one real root which is always positive. Under this condition, the truss will snap back and forth elastically after tensile yielding.

4. CASE B: LOW YIELD STRESS ($\alpha < 1$)

For $\alpha < 1$, the truss may yield in compression. When compressive yielding starts, the height of the truss y_{yc} is given by equation (10). If $y_{yc} \geq y_{c1}$, compressive yielding may occur before the elastic snap-through which is determined by the sign of f_1. Denote the velocity at $y = y_{yc}$ by $\dot{y}_{yc}$. From equations (10) and (11), we obtain

$$\dot{y}_{yc}^2 = v^2 - \tfrac{1}{2}\alpha^4 + 2p[1-(1-\alpha^2)^{\frac{1}{2}}] = f_4. \tag{32}$$

If $f_4 \leq 0$, the truss will move backward before compressive yielding occurs. In this case, the truss can only oscillate elastically without snap-through. On the other hand, if $f_4 > 0$, the tress may yield in compression before snap-through.

After compressive yielding, the stress in the bar remains a constant value $(-\sigma_y)$. We can obtain the equation of motion by setting $\sigma = -\sigma_y$ in equation (2). Thus we have

$$\ddot{y} - \alpha^2 y + p = 0. \tag{33}$$

The energy equation during compressive yielding can be found from equations (33), (10) and (32). It is

$$\dot{y}^2 = v^2 + \tfrac{1}{2}\alpha^4 + \alpha^2(y^2-1) + 2p(1-y). \tag{34}$$

Putting $\dot{y} = 0$ in equation (34), we obtain

$$v^2 + \tfrac{1}{2}\alpha^4 + \alpha^2(y^2-1) + 2p(1-y) = 0 \tag{35}$$

which is a quadratic equation of y. The condition for equation (35) possessing a double root is

$$f_5 = (\alpha^2 - p)^2 - \alpha^2(\tfrac{1}{2}\alpha^4 + v^2) = 0. \tag{36}$$

This double root is

$$y_{c2} = p^*/\alpha^2, \tag{37}$$

where $p = p^*$ is the root of equation (36). The root $y = y_{c2}$ is also a saddle point in the phase–plane diagram. Note that equations (36) and (37) are valid only if $y_{yc} \geq y_{c2}$. From equations (10) and (37), it is found that the condition $y_{yc} \geq y_{c2}$ is equivalent to the following condition:

$$\alpha^2(1-\alpha^2)^{\frac{1}{2}} - p^* \geq 0, \tag{38}$$

where $p = p^*$ satisfies equation (36). Since the saddle point for elastic snap-through y_{c1} of equation (15) must satisfy the cubic equation

$$y_{c1}^3 - y_{c1} + p = 0, \tag{39}$$

where p satisfies equation (13), we know that the condition $y_{yc} \geq y_{c2}$ is equivalent to the condition for compressive yielding occurring before the elastic snap-through, viz. $y_{yc} \geq y_{c1}$. Under the condition (38), if

$$p \leq \alpha^2 - \alpha(\tfrac{1}{2}\alpha^4 + v^2)^{\frac{1}{2}}, \tag{40}$$

we have $f_5 \geq 0$ and equation (35) possesses two real roots of y. In this case, there is no snap-through and the truss will oscillate elastically after compressive yielding. On the other hand, if

$$p > \alpha^2 - \alpha(\tfrac{1}{2}\alpha^2 + v^2)^{\frac{1}{2}}, \tag{41}$$

then $f_5 < 0$ and equation (35) possesses no real roots and there is snap-through during compressive yielding. Let us denote the velocity at $y = 0$ by $\dot{y}_0$. From equation (34), we have

$$\dot{y}_0^2 = v^2 + \tfrac{1}{2}\alpha^4 - \alpha^2 + 2p. \tag{42}$$

The total strain at $y = 0$ is

$$\varepsilon_0 = -\frac{h_0^2}{2a^2}. \tag{43}$$

After passing through the point $y = 0$, the stress–strain relation becomes elastic and satisfies the equation

$$\sigma + \sigma_y = E(\varepsilon - \varepsilon_0). \tag{44}$$

From equations (1), (2), (43) and (44), we have the following equation of motion in the elastic range of deformation after compressive yielding:

$$\ddot{y} + y(y^2 - \alpha^2) + p = 0. \tag{45}$$

The energy equation can be obtained from equations (45) and (42). It is

$$\dot{y}^2 = v^2 - \alpha^2(1 - y^2) + 2p(1 - y) + \tfrac{1}{2}(\alpha^4 - y^4). \tag{46}$$

The stress in the bar will change its sign in this range of deformation. When the stress becomes tensile and reaches the yield limit, tensile yielding occurs. At this moment, the height of the truss is y_{yt} which can be obtained from equation (44) by setting $\sigma = \sigma_y$. Thus we obtain

$$y_{yt} = -(2)^{\frac{1}{2}}\alpha. \tag{47}$$

The velocity at $y = y_{yt}$ can be found from equation (46). It satisfies

$$\dot{y}_{yt}^2 = v^2 - \alpha^2 + \tfrac{1}{2}\alpha^4 + 2p[1 + (2)^{\frac{1}{2}}\alpha]. \tag{48}$$

It can be shown that equation (48) is always positive under the condition of (41); i.e. tensile yielding is always present after snap-through.

During tensile yielding, the stress in the bar remains a constant value $\sigma = \sigma_y$. From equation (2), we can derive the equation of motion which is identical to equation (19). The corresponding energy equation can be obtained from equations (19), (47) and (48),

$$\dot{y}^2 = v^2 - \alpha^2(1 + y^2) + 2p(1 - y) + \tfrac{5}{2}\alpha^4. \tag{49}$$

By setting $\dot{y} = 0$ in equation (49), we obtain the minimum height of the truss y_m

$$y_m = -\frac{p}{\alpha^2} - \left[\left(\frac{p}{\alpha^2} + 1\right)^2 + \frac{v^2}{\alpha^2} + \frac{5}{2}\alpha^2 - 2\right]^{\frac{1}{2}}. \tag{50}$$

At this height, the total strain ε_m is given by equation (22). After the point $y = y_m$ is reached, the truss will move upward and the elastic stress–strain relation in the backward motion is given by equation (23). Hence the equation of motion and the energy equation for the upward motion of the truss are identical with equations (24) and (25) in which y_m is given by equation (50). Equation (28) also holds for the case $\alpha < 1$. Since $y_m < -(2)^{\frac{1}{2}}\alpha$, $\dot{y}^*_{yc}$ is complex. This shows that the truss will move downward again before the compressive yield point is reached. In other words, after tensile yielding, the truss can only oscillate elastically in its snapped configuration.

The foregoing discussion holds only when the condition (38) is valid. Suppose that

$$\alpha^2(1-\alpha^2)^{\frac{1}{2}} - p^* < 0, \tag{51}$$

then $y_{yc} < y_{c1}$. In this case, the critical condition for snap-through is dominated by the elastic condition, equation (13). The compressive yielding is always present during snap-through. The post-buckling motion of the truss is exactly the same as that discussed previously in this section.

In the following, we shall consider two special cases, namely, dynamic buckling due to a step loading ($v = 0$) and dynamic buckling due to an impulsive loading ($p = 0$).

5. STEP LOADING

In the case of step loading, we can set $v = 0$ in all preceding equations. Let us consider the following cases of α:

(i) $\alpha \geq 1$

In this case, there is no possibility for compressive yielding. From equations (13) and (15) we can find the critical load and the critical height for the elastic snap-through:

$$p_c = \tfrac{8}{27} \tag{52}$$

and

$$y_{c1} = \tfrac{1}{3}. \tag{53}$$

When $p < \frac{8}{27}$, there is no possibility for snap-through and the truss will oscillate elastically. When $p = \frac{8}{27}$, the height of the deformed truss will approach $y = \frac{1}{3}$ asymptotically. The condition for tensile yielding is determined by the sign of f_2 in equation (18). If

$$\tfrac{8}{27} < p \leq \tfrac{1}{4}\alpha^2[(1+\alpha^2)^{\frac{3}{2}} - 1] \tag{54}$$

there is no tensile yielding and the truss will snap back and forth elastically. Note that the condition (54) does not exist if $1 \leq \alpha \leq \frac{4}{3}$. When α falls within this range, there is always tensile yielding after snap-through.

Next, let us consider the motion of the truss after the tensile yielding. The minimum height of the truss, y_m, is determined by equation (21),

$$y_m = -\frac{p}{\alpha^2} - \left[\left(\frac{p}{\alpha^2}+1\right)^2 + \frac{\alpha^2}{2}\right]^{\frac{1}{2}}. \tag{55}$$

The critical load p_c for the backward snap-through must satisfy equation (30) which can be expressed as

$$27p_c^2 + 2y_m(9\alpha^2 - 4y_m^2)p_c + \alpha^4(2\alpha^2 - y_m^2) = 0. \tag{56}$$

The saddle point $y = y_{c2}$ will be found from equation (31). For

$$p_c > p > \tfrac{1}{4}\alpha^2[(1+\alpha^2)^{\frac{3}{2}} - 1] \tag{57}$$

the truss will snap backward after tensile yielding and then oscillates elastically. When $p = p_c$, the truss will finally stop at the height $y = y_{c2}$ in the upward motion. When $p > p_c$, after tensile yielding, the truss can only oscillate elastically without the appearance of the backward snap-through. The maximum height of the truss during the post-buckling oscillation is determined by equation (29).

The trajectories for $\alpha = 2$ are shown by curves (a)–(f) in Fig. 4 in the order of ascending values of p. The black circles on the curves represent the position where tensile yielding initiates.

(ii) $2(2)^{\frac{1}{2}}/3 < \alpha < 1$

In this case we can find p^* from equation (36) as

$$p^* = \alpha^2[1-\alpha/(2)^{\frac{1}{2}}]. \tag{58}$$

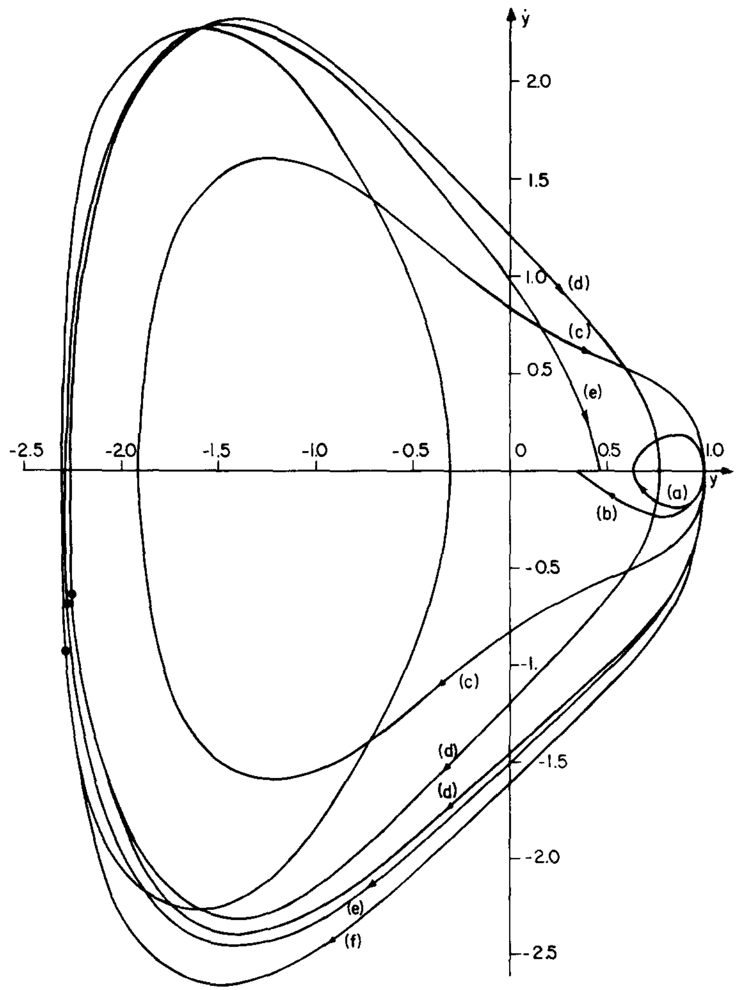

FIG. 4. Phase–plane diagram for $\alpha = 2$, $v = 0$, and various values of p.

The inequality condition (51) is fulfilled hence the critical load is the elastic critical load. When $p < \frac{8}{27}$, there is no possibility for snap-through and the truss will oscillate elastically. When $p = \frac{8}{27}$, the truss will deform to a height $y = \frac{1}{3}$ asymptotically. When $p > \frac{8}{27}$, snap-through will occur. Compressive yielding and tensile yielding are found during the motion of snap-through. After tensile yielding the truss cannot snap backward but oscillates elastically in its snapped configuration.

(iii) $\alpha \leq 2(2)^{\frac{1}{2}}/3$

In this case, the inequality condition (38) is satisfied. The critical load for compressive yielding can be found from equation (32). Therefore if $p \leq \frac{1}{4}\alpha^2[1+(1-\alpha^2)^{\frac{1}{2}}]$, there is neither compressive yielding nor snap-through and the truss can only oscillate elastically. The critical load for snap-through during compressive yielding can be found from equation (36),

$$\bar{p}_c = \alpha^2[1-\alpha/(2)^{\frac{1}{2}}]. \tag{59}$$

Hence if $\frac{1}{4}\alpha^2[1+(1-\alpha^2)^{\frac{1}{2}}] < p < \bar{p}_c$, there is compressive yielding but no snap-through. The saddle point is given by equation (37). It is found to be

$$y_{c2} = 1-\alpha/(2)^{\frac{1}{2}}. \tag{60}$$

When $p = \bar{p}_c$, the height of the truss will approach this saddle point asymptotically. When $p > \bar{p}_c$, there is snap-through during compressive yielding.

Tensile yielding will occur after snap-through. After tensile yielding, the truss cannot snap backward but oscillates elastically in its snapped configuration. The minimum height y_m and the maximum height of the truss during the post-buckling oscillation are determined by equations (55) and (29), respectively.

The trajectories of the motion of the truss for $\alpha = 0{\cdot}75$ are shown by curves (*a*)–(*d*) in Fig. 5 in the order of ascending values of p. The white circles in Fig. 5 represent the positions for the initiation of compressive yielding and the black circle shows the initiation of tensile yielding.

The results of the foregoing analysis for the case of step loading ($v = 0$) are shown graphically by the p–α diagram in Fig. 6. There are six regions separated by solid lines. For any values of α and p, we can find a point in these regions. Information about yielding and snap-through for any values of α and p is shown in the diagram.

6. IMPULSIVE LOADING

Under impulsive loading, the initial velocity of the truss is prescribed but there is no load acting on the truss at $t > 0$. In this case, $v \neq 0$ and $p = 0$. We shall discuss the snap-through behavior in reference to whether $\alpha \geq 1$ or $\alpha < 1$.

(i) $\alpha \geq 1$

There is no compressive yielding for $\alpha \geq 1$. The critical velocity for elastic snap-through is found from equation (13). It is

$$v_c = (2)^{-\frac{1}{2}}, \tag{61}$$

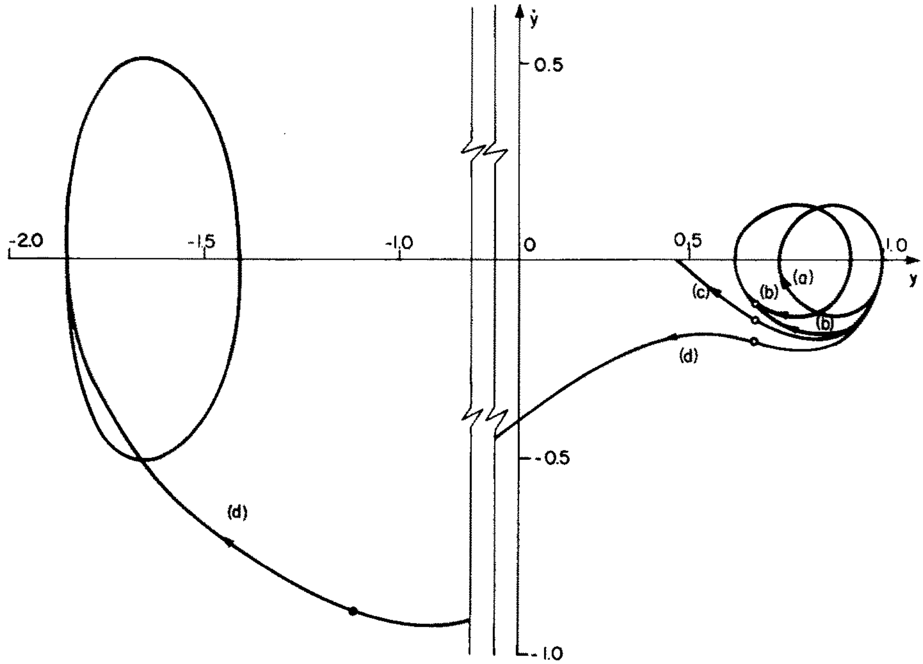

FIG. 5. Phase–plane diagram for $\alpha = 0{\cdot}75$, $v = 0$ and various values of p.

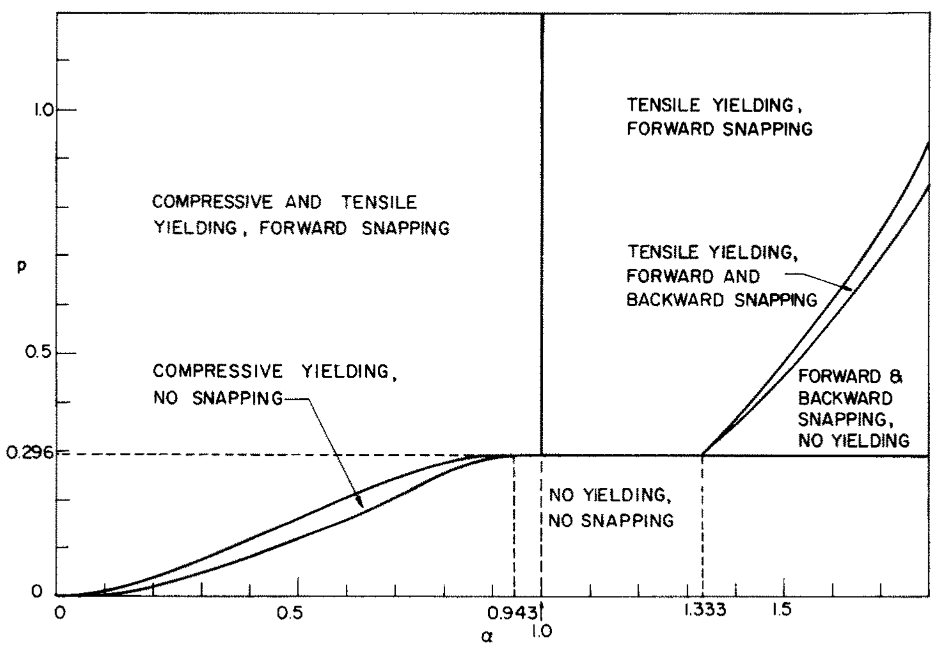

FIG. 6. p–α diagram for the case of step loading ($v = 0$).

and the saddle point is given by equation (15),

$$y_{c1} = 0. \tag{62}$$

Thus, for $v < v_c$ there is no snap-through and for $v = v_c$ the truss will approach a flat position $y = 0$ asymptotically. The critical condition for tensile yielding after snap-through is given by equation (18). Hence for $(2)^{-\frac{1}{2}} < v \leq \alpha^2/(2)^{\frac{1}{2}}$, there is no tensile yielding but the truss can snap back and forth elastically.

After tensile yielding the truss moves upward. The minimum height of the truss is given by equation (21),

$$y_m = -\left(1+\frac{v^2}{\alpha^2}+\frac{\alpha^2}{2}\right)^{\frac{1}{2}}. \tag{63}$$

In the upward motion of the truss, backward snap-through may occur. The critical initial velocity for backward snap-through, v_c^*, is determined by equations (30) and (63),

$$v_c^* = \alpha(\tfrac{3}{2}\alpha^2-1)^{\frac{1}{2}}. \tag{64}$$

The corresponding saddle point is found from equation (28),

$$y_{c2} = 0. \tag{65}$$

When $\alpha^2/(2)^{\frac{1}{2}} < v < v_c^*$, the truss will snap back and forth elastically after the tensile yielding. When $v = v_c^*$, the truss will approach the saddle point y_{c2} asymptotically. Finally, if $v > v_c$, after tensile yielding, the truss cannot snap backward but oscillates elastically in its snapped configuration. The maximum height of the truss in the post-buckling oscillation is determined by equation (29).

The trajectories for $\alpha = 1{\cdot}2$ are shown by curves (*a*)–(*f*) in Fig. 7 in the order of ascending values of v. Again, the positions for the initiation of tensile yielding are shown by black circles.

(ii) $\alpha < 1$

In the case of impulsive loading, we may consider $p = 0$ in equation (36). Thus $p^* = 0$. The inequality condition (38) is always fulfilled. Accordingly, there is no possibility for elastic snap-through to occur before compressive yielding.

The critical initial velocity for compressive yielding is determined by equation (32). For $v \leq (2)^{\frac{1}{2}}\alpha^2/2$, there is no compressive yielding and the truss oscillates elastically in its unsnapped configuration. The critical initial velocity for snap-through during compressive yielding is determined by equation (36) and the saddle point is found at $y_{c2} = 0$ from equation (37). Hence for $\alpha^2/(2)^{\frac{1}{2}} < v < \alpha(1-\frac{1}{2}\alpha^2)^{\frac{1}{2}}$ there is no snap-through and the truss oscillates elastically after compressive yielding. When $v = \alpha(1-\frac{1}{2}\alpha^2)^{\frac{1}{2}}$, the height of the truss will approach the saddle point $y = 0$ asymptotically during compressive yielding. When $v > \alpha(1-\frac{1}{2}\alpha^2)^{\frac{1}{2}}$, snap-through occurs during compressive yielding and tensile yielding follows. After tensile yielding, the truss will oscillate elastically without the appearance of the backward snap-through. The minimum height y_m and the maximum height of the truss during the post-buckling oscillation are given by equations (63) and (29), respectively.

In Fig. 8, the trajectories for $\alpha = 0{\cdot}8$ are shown by curves (*a*)–(*d*) in the order of ascending values of v. The compressive and tensile yield points in the phase–plane diagram are represented respectively by white and black circles. In a manner similar to the case of step loading,

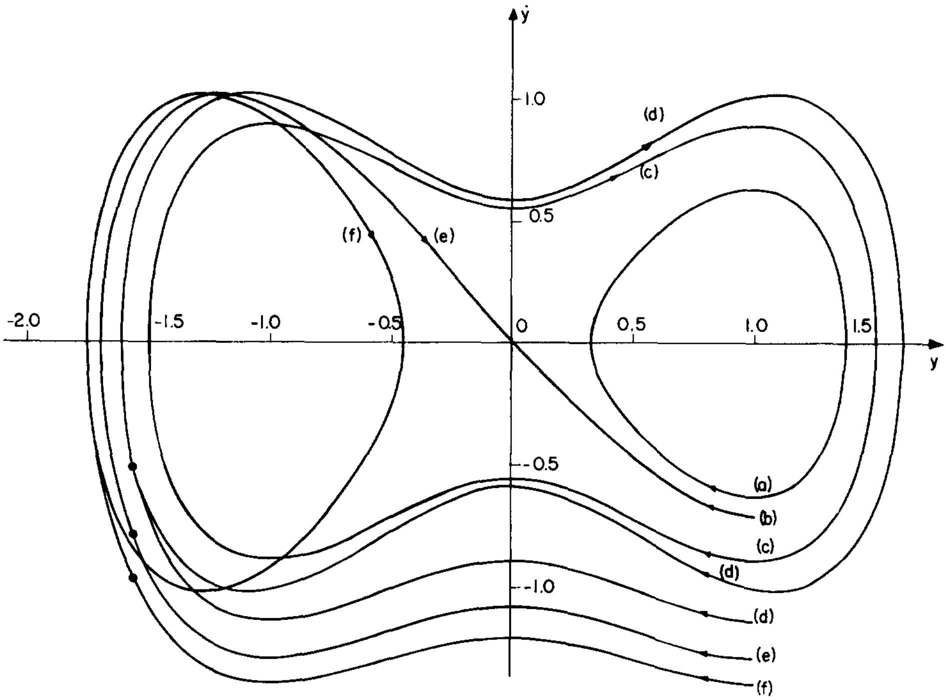

FIG. 7. Phase–plane diagram for $\alpha = 1{\cdot}2$, $p = 0$ and various values of v.

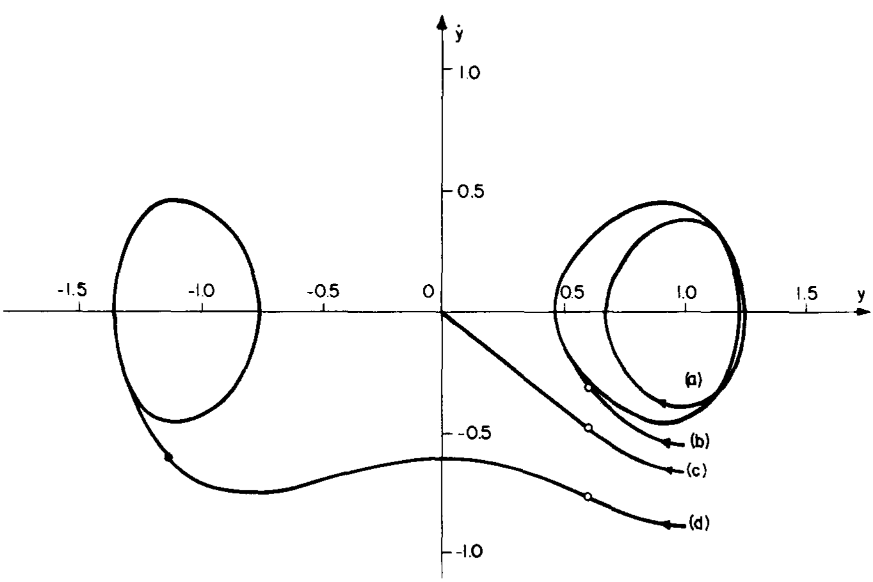

FIG. 8. Phase–plane diagram for $\alpha = 0{\cdot}8$, $p = 0$ and various values of v.

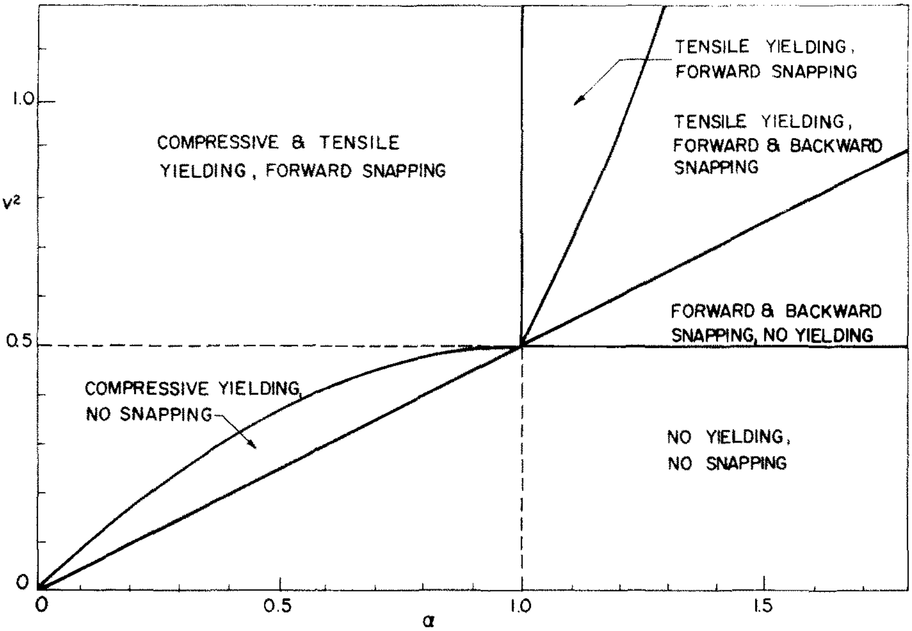

FIG. 9. v^2–α diagram for the case of impulsive loading ($p = 0$).

we can divide the v^2–α plane into several regions according to the characteristics of the truss in yielding and snap-through as shown in Fig. 9. Again, there are six regions. It is found that these six regions have a common vertex at the point $\alpha = 1$ and $v^2 = 0{\cdot}5$. However, this common vertex does not exist in the p–α diagram (Fig. 6) for the case of step loading.

7. CONCLUSIONS

The following conclusions on the elastic–plastic dynamic snap-through of the simple truss can be made directly from Figs. 6 and 9.

(1) In the case of low values of yield stress, the dynamic snap-through may occur after compressive yielding. The critical load for snap-through increases with the yield stress. Tensile yielding is always present after snap-through. The truss cannot snap back after tensile yielding but oscillates elastically in its snapped configuration.

(2) In the case of intermediate values of yield stress, snap-through is essentially governed by the elastic action. In the case of step loading, there is no possibility for the backward snap-through after tensile yielding. However, in the case of impulsive loading, the backward snap-through may occur when the applied impulse is sufficiently small. Under large impulse, the plastic strain due to tensile yielding becomes significant and the truss cannot snap backward.

(3) In the case of high values of yield stress, the dynamic snap-through is elastic. The truss may snap back and forth if the applied load is sufficiently small. On the other hand, if the applied load is large, the truss can only snap forward and then oscillates elastically in its snapped configuration after tensile yielding.

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Абстракт—Исследуется эффект пластической деформации на поведение динамического Выпучивания пологих конструкций, для случая обыкновенной фермы, состоящей с типичных стержней, с массой приложенной и серединному шарниру. Предполагается, что материал фермы является упруго-идеально пластическим, с равными пределами пластичности для растяжения и сжатия. Нагрузка приложенна к серединному шарниру. Констатируется два типа нагрузки, именно, ступенчатая и импульсивная. Находится, что для материалов с низким уровнем напряжения течения, нагрузка динамического выпучивания повышается с напряжением течения. Тем не менее, для материалов с высоким уровнем напряжения течения, явление динамического выпучивания преобледается над упругим действием. Когда приложенная нагрузка является достаточно большая, прощелкивание фермы может быть предупреждено растятгивающим течением, присумствующим деформации выпучивания фермы за пределом упругости.